Distribution Theory I

1.1. The Space $\mathcal{D}_K$.

Let $K \subset \mathbb{R}^n$ be compact.

$$\mathcal{D}_K = \{ \varphi : \varphi \in C^\infty(\mathbb{R}^n),\ \operatorname{spt}\varphi \subset K \} \text{ where}$$

$$\operatorname{spt}\varphi = \overline{\{ x : \varphi(x) \neq 0 \}}.$$

$\mathcal{D}_K$ forms a topological vector space with topology induced from the metric

$$d(\varphi, \psi) = \sum_{m=0}^{\infty} \frac{1}{2^m} \frac{\|\varphi - \psi\|_m}{1 + \|\varphi - \psi\|_m}, \text{ where}$$

$$\|\varphi\|_m = \sup_K \{ |D^\alpha \varphi(x)| : 0 \leq |\alpha| \leq m \}.$$

In this metric, $\varphi_j \to \varphi$ iff $\|\varphi_j - \varphi\|_m \to 0$ $\forall m$, that is,

$$D^\alpha \varphi_j \rightrightarrows D^\alpha \varphi \text{ in } K,\ \forall \alpha.$$

1.2. The Space of Test Functions $\mathcal{D}(U)$.

Let $U \subseteq \mathbb{R}^n$ be open and

$\mathcal{D}(U) = C_c(U)$ the vspace of all smooth functions with cpt spt in U endowed with a topology such that

$\varphi_j \to \varphi$ in $\mathcal{D}(U)$ iff $\exists$ cpt $K \subset U$ s.t. $\operatorname{spt}\varphi_j \subset U$

$\forall j$, and $\varphi_j \to \varphi$ in $\mathcal{D}_K$.

Please refer to [R2] for a description of the topology on $\mathcal{D}(U)$. In our development, the knowledge of the convergence of $\{\varphi_j\}$ is sufficient for most purposes.

1.3. The Space of Distributions

A distribution is a linear functional on $\mathcal{D}(U)$ which satisfies

$$\Lambda(\varphi_j) \to 0 \quad \text{for } \forall \varphi_j \in \mathcal{D}(U),\ \varphi_j \to 0, \text{ in } \mathcal{D}(U)$$

Note that it also means

$$\Lambda \varphi_j \to \Lambda \varphi \quad \text{for } \varphi_j, \varphi \in \mathcal{D}(U),\ \varphi_j \to \varphi \text{ in } \mathcal{D}(U).$$

Example 1 $\quad f \in L^1_{loc}(U)$. Define $\Lambda = \Lambda_f$ by

$$\Lambda \varphi = \int f \varphi \, d\mathcal{L}^n, \quad \varphi \in \mathcal{D}(U).$$

It is clearly that $\Lambda \in \mathcal{D}'(U)$. In fact, $\Lambda_f = \Lambda_g$ iff $f = g$ ae. So $f \mapsto \Lambda_f$ is injective. Every loc. integrable function is a distribution.

Example 2 $\quad \mu$ Radon measure in $\mathbb{R}^n$. Define $\Lambda = \Lambda_\mu$

$$\Lambda \varphi = \int \varphi \, d\mu, \quad \varphi \in \mathcal{D}(\mathbb{R}^n)$$

again Λ is a distribution. Moreover, $\Lambda_\mu = \Lambda_\nu$ iff $\mu = \nu$.
Thus every Radon measure is a distribution in $\mathbb{R}^n$.

Example 3 The delta function. Let $x_0 \in U$ and define

$$\delta_{x_0}(\varphi) = \varphi(x_0), \quad \forall \varphi \in \mathcal{D}(U).$$

δ_{x_0} is a distribution.

We note the structure of $\mathcal{D}'(U)$:

- $\mathcal{D}'(U)$ is a vector space over $\mathbb{R}$ or $\mathbb{C}$,
- $f \in C^\infty(U)$, $\Lambda \in \mathcal{D}'(U)$ then

$$(f\Lambda)(\varphi) \stackrel{\text{def}}{=} \Lambda(f\varphi), \quad \varphi \in \mathcal{D}(U),$$

is a distribution. So, the product of a (smooth) function with a distribution is a distribution. (But, there is no definition for the product of two distributions.)

- For $\Lambda \in \mathcal{D}'(U)$ and D^α,

$$(D^\alpha \Lambda)(\varphi) \stackrel{\text{def}}{=} (-1)^{|\alpha|} \Lambda(D^\alpha \varphi), \quad \varphi \in \mathcal{D}(U),$$

is a distribution. So, you can always differentiate distribution

$\boxed{\text{Notation}}$ $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j \in \mathbb{N}$, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$.

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}, \quad D_j = \frac{\partial}{\partial x_j}.$$

$$\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$$

$\alpha \leq \beta$ if $\alpha_j \leq \beta_j$ all j.

Example 4 Let $h(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$ then

$$\int_{-\infty}^{\infty} h\varphi' = \int_{0}^{\infty} h\varphi' = h\varphi\Big|_{0}^{\infty} - \int_{0}^{\infty} h'\varphi = -h(0)\varphi(0).$$

therefore, $Dh = \delta_0$.

1.4. the Support of a Distribution

Proposition 1.1 $\Lambda \in \mathcal{D}'(U)$ iff $\forall K \subset U$, $\exists N$ and C s.t.

$$|\Lambda\varphi| \leq C\|\varphi\|_N, \quad \forall \varphi \in \mathcal{D}_K.$$

Proof: $(\Leftarrow)$ Let $\varphi_j \to 0$ in $\mathcal{D}(U)$. then $\exists K$ cpt in U s.t. φ_j spted in K and $\varphi_j \to 0$ in $\mathcal{D}_K$. The latter means that $\|\varphi_j\|_m \to 0$ $\forall m$. In particular, taking $m = N$ and from $|\Lambda\varphi_j| \leq C\|\varphi_j\|_N$ we deduce that

$$\Lambda\varphi_j \to 0 \quad \text{as } \|\varphi_j\|_N \to 0.$$

So, $\Lambda \in \mathcal{D}'(U)$.

$(\Leftarrow)$ Suppose on the contrary $\exists K$ cpt in U such that $\forall N$,

$$\sup_{\substack{\varphi \neq 0 \\ \mathcal{D}_K}} \frac{|\Lambda\varphi|}{\|\varphi\|_N} = \infty$$

For each N, choose $\varphi_N \in \mathcal{D}_K$ s.t.

$$\frac{|\Lambda \varphi_N|}{\|\varphi_N\|_N} \geq N .$$

Let

$$\psi_N = \frac{\varphi_N}{N\|\varphi_N\|_N} \in \mathcal{D}_K$$

then

$$\|\psi_N\|_N = \frac{1}{N} .$$

We claim that $\psi_N \to 0$ in $\mathcal{D}_K$, ie, $\forall m$,

$$\|\psi_N\|_m \to 0 \quad \text{as } N \to \infty .$$

For, for a fixed m, $\forall N \geq m$,

$$\|\psi_N\|_m \leq \|\psi_N\|_N = \frac{1}{N} \to 0 \quad \text{as } N \to \infty .$$

Claim holds. But then

$$|\Lambda \psi_N| = \left|\Lambda\left(\frac{\varphi_N}{N\|\varphi_N\|_N}\right)\right| \geq 1$$

that means $\Lambda \psi_N \not\to 0$, Λ can't be a distibution

Contiadiction holds. Hence, must $\exists N, C$ s.t.

$$|\Lambda \varphi| \leq C\|\varphi\|_N, \quad \forall \varphi \in \mathcal{D}_K .$$

Let $V \subset U$ open. $\Lambda \in \mathcal{D}'(U)$ is called to <u>vanish</u> on V

if $\Lambda\varphi = 0$, $\forall \varphi$ spt in V. Let

$$W = \bigcup \{ V : \Lambda \text{ is vanishing in } V \}$$

<u>Proposition 1.2</u> Λ <u>vanishes on</u> W.

Proof. $W = \bigcup V$, let $\{\varphi_j\}$ be a partition of unity

subordinate to V, ie, $\exists V_j$ s.t. spt $\varphi_j \subset\subset V_j$ and

$\sum \varphi_j(x) = 1$, $x \in W$. Furthermore, for each $K \subset\subset W$,

$\exists \varphi_1, \ldots, \varphi_m$ s.t. $\displaystyle\sum_{j=1}^{m} \varphi_j(x) = 1$, $\forall x \in K$. (locally finite)

see [R1] or [R2] for a construction.

Now, let $\varphi \in \mathcal{D}(U)$, spt $\varphi \subset\subset W$. We'd like to show

$\Lambda\varphi = 0$.

Note that

$$\varphi(x) = \sum_{j=1}^{m} \varphi(x)\varphi_j(x), \quad \forall x \in U.$$

(If $x \in W$, use $\sum_1^m \varphi_j(x) = 1$. If $x \notin W$, all $\varphi(x)$, $\varphi_j(x)$'s vanish.)

So

$$\Lambda\varphi = \Lambda\Big(\sum_1^m \varphi\varphi_j\Big) = \sum_1^m \Lambda(\varphi\varphi_j) = 0 \quad \because \text{spt } \varphi\varphi_j \subset\subset V_j.$$

\#

Define the <u>support</u> of Λ, S_Λ, to be

$$S_\Lambda = U \setminus W.$$

<u>Proposition 1.3</u> <u>Let</u> $\psi \in C^\infty(U)$, $\psi \equiv 1$ <u>on some open</u> G, $S_\Lambda \subset G$.

$$\psi \Lambda = \Lambda.$$

Proof. Need to show $\Lambda(\psi\varphi) = \Lambda\varphi$, $\forall \varphi \in \mathcal{D}(U)$.

Now, $\psi\varphi - \varphi = 0$ in G. So $\operatorname{spt}(\psi\varphi - \varphi) \subset \overline{U \setminus G} = U \setminus G$
$\subset W$

By the previous proposition

$$\Lambda(\psi\varphi - \varphi) = 0$$

$$\therefore \quad \Lambda(\psi\varphi) - \Lambda\varphi = 0, \quad \forall \varphi \in \mathcal{D}(U).$$

<u>Proposition 1.4</u> <u>Let</u> S_Λ <u>be compact</u> $\subset U$. <u>Then</u> $\exists\, N, C$ s.t.

$$|\Lambda\varphi| \leq C \|\varphi\|_N, \quad \forall \varphi \in \mathcal{D}(U)$$

Proof. Let G be open set s.t. $S_\Lambda \subset G$, $\overline{G} \subset U$, $\overline{G}$ compact.
Fix $\psi \in \mathcal{D}(U)$, $0 \leq \psi \leq 1$, $\psi \equiv 1$ on G. Then Prop 1.3

asserts

$$\Lambda\varphi = \Lambda(\psi\varphi), \quad \forall \varphi \in \mathcal{D}(U)$$

So $\operatorname{spt}(\psi\varphi) \subset K = \operatorname{spt}$ of $\psi \subset U$, by Prop 1.1. $\exists\, N, C$ s.t.

$$|\Lambda(\psi\varphi)| \leq C \|\psi\varphi\|_N.$$

Using the product rule $D(\psi\varphi) = \psi D\varphi + (D\psi)\varphi$ N-many times

we get
$$\|\psi\varphi\|_N \leq C\|\varphi\|_N,$$
where C depends on $\|\psi\|_N$. So
$$|\Lambda\varphi| = |\Lambda(\psi\varphi)| \leq C\|\varphi\|_N, \ \forall \varphi \in \mathscr{D}(U). \ \#$$

Example 1.5 Let $\delta = \delta_{x_0}$, $x_0 \in U$. We claim
$$\operatorname{spt} D^\alpha \delta = \{x_0\}$$
for any α. For, let $x \neq x_0$ and $B_\rho(x) \not\ni x_0$, then for φ spt in $B_\rho(x)$, $D^\alpha\varphi$ also spt in $B_\rho(x)$, so clearly δ and $D^\alpha\delta$ vanishes on $B_\rho(x)$. that's, $B_\rho(x) \subset W$. It shows that $W \supset U\setminus\{x_0\}$. On the other hand, if $x_0 \in W$, then $\operatorname{spt} D^\alpha\delta$ is empty which implies that $(D^\alpha\delta)(\varphi) = 0$ for all $\varphi \in \mathscr{D}(U)$.

WLOG let $x_0 = 0$. Fix some $\varphi \in \mathscr{D}(U)$, $\varphi(0) = 1$ and consider the fcn $\psi(x) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}\varphi(x) \in \mathscr{D}(U)$. Then
$$\begin{aligned} 0 = (D^\alpha\delta)(\psi) &= (-1)^{|\alpha|} D^\alpha(x_1^{\alpha_1} \cdots x_n^{\alpha_n}\varphi(x))\Big|_{x=0} \\ &= (-1)^{|\alpha|}\alpha_1! \cdots \alpha_n!\,\varphi(0) \\ &\neq 0, \text{ contradiction.} \end{aligned}$$
$\therefore W = U\setminus\{x_0\}$.

Theorem 1.5 Let S_Λ be $\{x_0\}$. Then $\exists N$ s.t.
$$\Lambda = \sum c_\alpha D^\alpha\delta, \quad \delta = \delta_{x_0}.$$

Take $x_0 = 0$.

Proof. By Prop. 1.4, $\exists\, N, C$ s.t.

$$|\Lambda\varphi| \le C\|\varphi\|_N, \quad \forall \varphi \in \mathcal{D}(U)$$

Let $\varphi \in \mathcal{D}(U)$ satisfy

$$D^\alpha\varphi(0) = 0, \quad |\alpha| \le N.$$

Claim: $\quad \Lambda\varphi = 0.$

First, by Taylor Expansion

$$\varphi(x) = \varphi(0) + \cdots + \sum_{|\alpha|=N} \frac{D^\alpha\varphi(c)}{\alpha!} x^\alpha, \quad c \text{ some mean-value.}$$

$$= \sum_{|\alpha|=N} \frac{D^\alpha\varphi(c)}{\alpha!} x^\alpha$$

As $D^\alpha\varphi(c) \to 0$ when $x \to 0$, given $\varepsilon > 0$, $\exists\, \delta$ st

$$|\varphi(x)| \le C\varepsilon |x|^N, \quad x \in B_\delta(0).$$

$$D_j\varphi(x) = D_j\varphi(0) + \cdots + \sum_{|\alpha|=N-1} \frac{D^\alpha D_j\varphi(c)}{(\alpha-1)!} x^\alpha$$

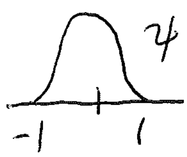
as before

$$|D_j\varphi(x)| \le C\varepsilon |x|^{N-1}, \quad x \in B_\delta(0).$$

In general, we get

$$|D^\beta\varphi(x)| \le C\varepsilon |x|^{N-|\beta|}, \quad x \in B_\delta(0), \ 0 \le |\beta| \le N.$$

Next, fix a bump function ψ and



set $\psi_r(x) = \psi(\frac{x}{r})$. $\psi_r\varphi$ spt in $B_r(0)$

and $\quad |D^\alpha(\psi_r\varphi)| = \left|\sum_{\beta+\gamma=\alpha} c_{\beta\gamma} D^\beta\psi_r D^\gamma\varphi\right| \le \sum |c_{\beta\gamma}|\, |x|^{-|\beta|}\, |D^\beta\psi(\frac{x}{r})|\, C\varepsilon |x|^{N-|\gamma|}$

$$\leq C\varepsilon|x|^{N-|\alpha|}$$

therefore,

$$|\Lambda(\psi_r\varphi)| \leq C\,\|\varphi\|_N$$
$$\leq C'\varepsilon$$

Letting $\varepsilon \to 0$, we conclude from Prop. 1.3

$$|\Lambda\varphi| = |\Lambda(\psi_r\varphi)| = 0 .$$

The claim is established. Now the theorem follows from the following linear algebra lemma.

Lemma Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_m$, Λ be linear fls on the vector space X. Let

$$\mathcal{N} = \{x : \Lambda_j x = 0,\ j = 1, \ldots, m\}.$$

Suppose that $\Lambda x = 0 \ \forall x \in \mathcal{N}$, that

$$\Lambda = \sum_{j=1}^{m} \alpha_j \Lambda_j \quad \text{for some } \alpha_1, \ldots, \alpha_m .$$

([R2] Lemma 3.9.)